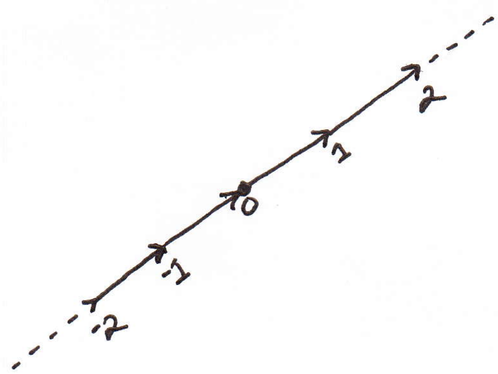
(1.1)

Much of what is studied in calculus is about space. Space and time, which classical mechanics regards as particular examples of Eucledian space, form the mathematical framework for expressing the laws that govern the interactions of objects in the physical world. In this section we give an intuitive introduction to the mathematical notion of space.

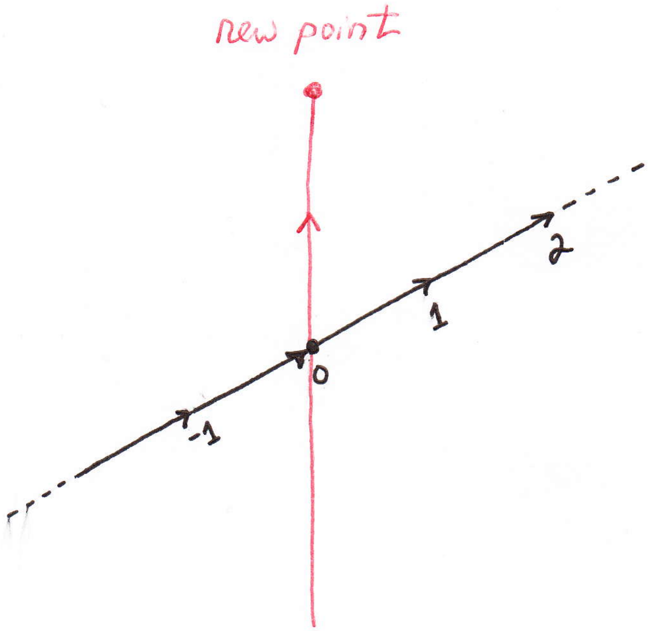
Linear Space

Imagine an empty space that is suddenly populated by one particle. Mathematically, we may conceive of this space as the collection of all points, where the particle may pop-up into existance. Suppose that the particle is traveling through space at constant speed that is measured with a stop watch. The set of all points in space visited by the particle is the directed line below



where negative numbers correspond to points visited by the particle in the past, positive numbers indicate points that will be visited in the future, and 0 is the location of the particle in the present moment.

If, in this manner, the particle visits all points in space, the geometry of the space is that of a number line and each point may be uniquely paired with a real number that indicates its address. Such space is completely described by one parameter, namely the real number, and is therefore said to be a one dimensional space.
If our space has a point not on the number line, by changing the direction of motion of the particle, we can make it travel from its current location to that new point.

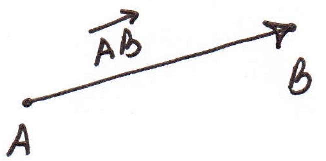


As we shall see later in this section, a space in which two non-parallel lines may be traced requires at least two parameters to indicate the address of a point. Such space is at least two dimensional. For now, observe that whenever two points are part of the space, then so are the points that lie on the line which passes through these two points. We make the following definition:

Def: A linear space is a collection of points such that whenever two points are in the collection, the set of points on the line which passes through these two points is also part of the collection.

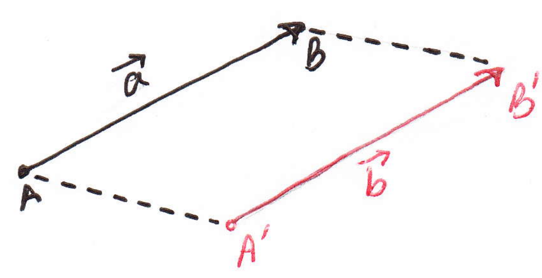
Vectors

The simplest motion from point A to point B in a linear space is along a line segment connecting these points. We represent the displacement from A to B by an arrow, whose tail is at A and with the arrowhead at B.



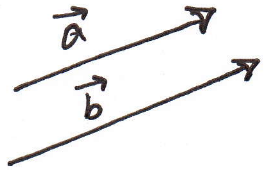
Def. A vector is a line segment with direction. We will symbolize vectors by boldface letters such as $\mathbf{a}$ or by drawing an arrow over the letter $\vec{a}$. When units of length are specified, we may denote the magnitude of a given vector by the symbol $\|\cdot\|$. Thus $\|\vec{a}\|$ refers to the length of the directed line segment $\vec{a}$.

A vector is a relationship between two points in linear space; if $\vec{a}$ is a vector from point A to point B and $\vec{b}$ is a vector from point A′ to B′, then $\vec{a} = \vec{b}$ iff the arrowhead of vector $\vec{b}$ is carried from B′ to B as soon as the tail of vector $\vec{b}$ is carried from A′ to A.



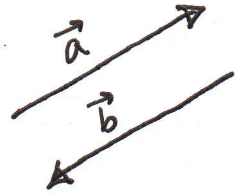
Stated differently, two vectors are equal iff they have the same magnitude and direction.

Ex. 1)



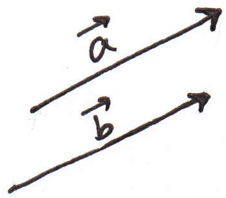
$\vec{a} \neq \vec{b}$ (why?)

2)



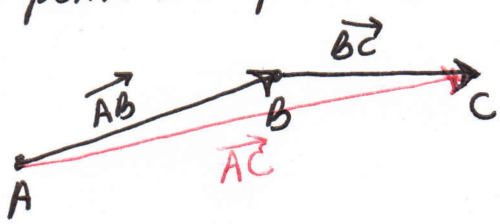
$\vec{a} \neq \vec{b}$ (why?)

3)



$\vec{a} = \vec{b}$ (why?)

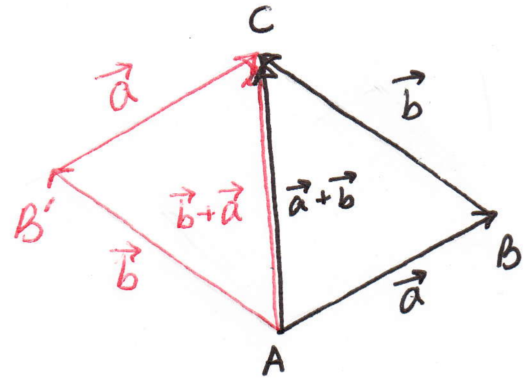
If a particle travels initially from point A to point B and then from B to point C, the total displacement is the directed line segment from point A to point C.



This motivates the following definition:

Def: The sum of two vectors $\vec{a}$ & $\vec{b}$, denoted $\vec{a}+\vec{b}$, is the vector obtained by first translating $\vec{b}$ so that its tail lies at the head of $\vec{a}$ and then drawing the vector with tail at the tail of $\vec{a}$ and head at the head of the translated $\vec{b}$.

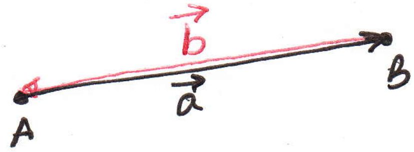
Given two vectors $\vec{a}$ and $\vec{b}$, it is not immediately obvious that $\vec{a}+\vec{b} = \vec{b}+\vec{a}$. As can be seen by the diagram below, a particle initially at point A that displaces through $\vec{a}$ followed by $\vec{b}$ will not, in general, pass through the same set of points as those visited by displacing through vector $\vec{b}$ followed by $\vec{a}$. However, the initial and final points of the trip are the same in both cases.



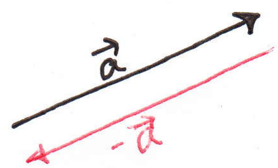
whether we displace from point A through the vector $\vec{a}+\vec{b}$ or through the vector $\vec{b}+\vec{a}$, we arrive at point C. This proves that $\vec{a}+\vec{b}=\vec{b}+\vec{a}$. For obvious reasons, the commutativity of vector addition is sometimes called "parallelogram law".

To make operations on vectors into an algebra, it is necessary to define the zero vector. The vector from point A to itself has no magnitude and is denoted $\vec{0}$. Since it is impossible to assign a direction to a line segment with no length, $\vec{0}$ is said to be directionless.

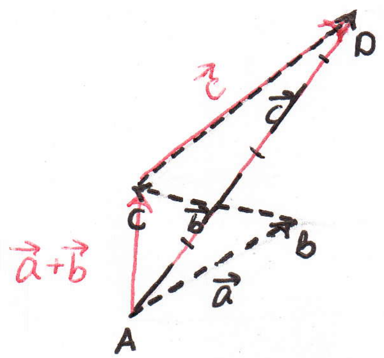
For any vector $\vec{a}$, there is some vector $\vec{b}$ such that $\vec{a}+\vec{b}=\vec{0}$; if $\vec{a}$ is a displacement from point A to point B and $\vec{b}$ is a displacement from point B to point A, the rules of vector addition dictate that $\vec{a}+\vec{b}$ is a displacement from A to A and $\vec{b}+\vec{a}$ is a displacement from B to B. Thus $\vec{a}+\vec{b}=\vec{b}+\vec{a}=\vec{0}$



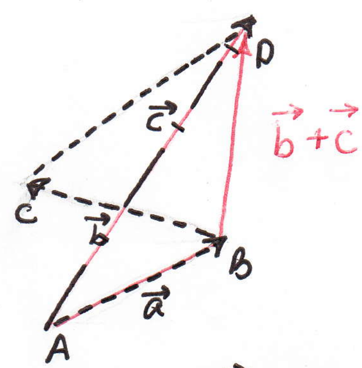
Def: Let $\vec{a}$ be a vector. By $-\vec{a}$ we denote the vector that has the same magnitude as $\vec{a}$, but opposite direction.



If $\vec{b}$ is another vector, then $\vec{b}-\vec{a}$ is the vector $\vec{b}+(-\vec{a}) = (-\vec{a})+\vec{b}$. We can extend the definition of vector addition to the case of several vectors. For example, the sum $\vec{a}+\vec{b}+\vec{c}$ may be interpreted as $(\vec{a}+\vec{b})+\vec{c}$ or as $\vec{a}+(\vec{b}+\vec{c})$. In both cases, the resulting vector is the same as seen in the diagram below:

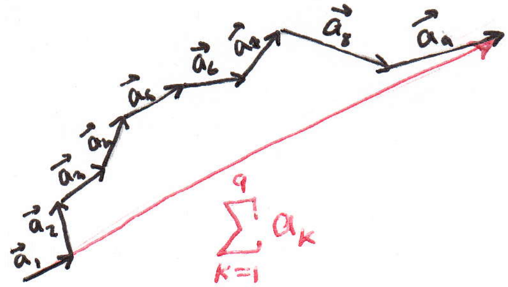


$(\vec{a}+\vec{b})+\vec{c} = \overrightarrow{AD}$

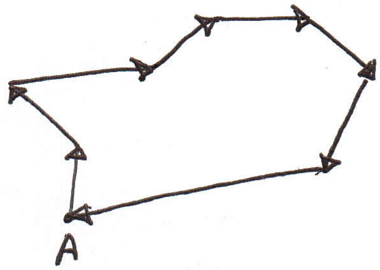


$\vec{a}+(\vec{b}+\vec{c}) = \overrightarrow{AD}$

Thus, vector addition is associative. The sum is the vector obtained by drawing a line segment from the first vector in the sum to the last vector. For example



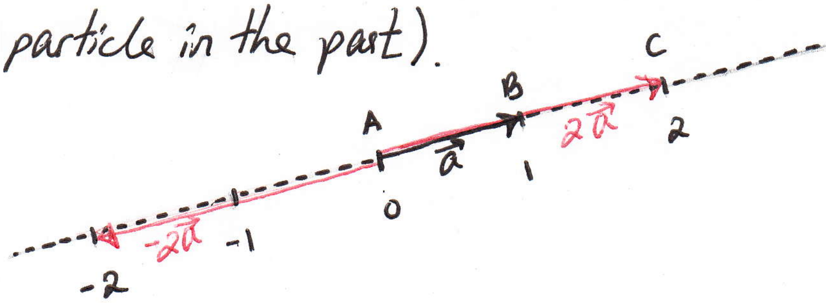
Comprehension Check



What is the sum of these vectors?

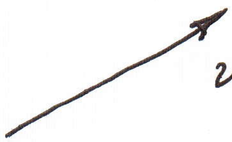
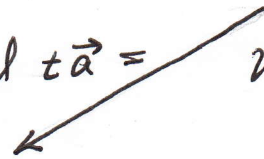
Answer

The sum is $\vec{0}$.
What can be said about vector multiplication? The notion of dot and cross products of the vectors $\vec{a}, \vec{b}$, $\vec{a}\cdot\vec{b}$ and $\vec{a}\times\vec{b}$ will be explored in later sections. In the present moment, let us motivate the product of a scalar and a vector.
Suppose that we wish to track the motion of a particle with the help of a stop watch. Suppose further that the particle is moving with constant speed. If we began tracking the motion when the particle was visiting point A, a second later the particle is visiting some other point B and two seconds later, the particle will be visiting some further point C. Letting $\vec{a} = \vec{AB}$, it becomes natural to identify the vector $\vec{AC}$ with $2\vec{a}$, as C is the point that will be visited by the particle 2 seconds in the future. In general, if $t \in \mathbb{R}$ is a real number, $t\vec{a}$ is the vector whose tail lies at A and whose arrowhead lies on the point visited by the particle t seconds in the future. (If $t<0$, the arrowhead is at a point visited by the particle in the past).

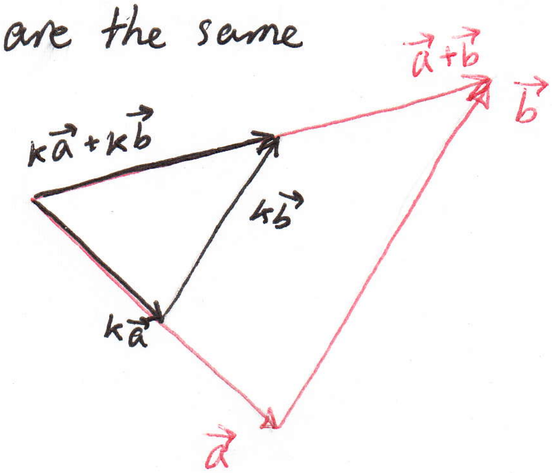


Def: Multiplication of a vector $\vec{a}$ by a scalar (i.e. real number) t produces a new vector $t\vec{a}$ obtained in the following way: $t\vec{a}$ is the vector of magnitude $|t|\,\|\vec{a}\|$ with the same direction as $\vec{a}$ if $t>0$, opposite direction to $\vec{a}$ if $t<0$, $\vec{0}$ if $t=0$. If $t\neq 0$, $\frac{\vec{a}}{t}$ represents $(\frac{1}{t})\vec{a}$.

Ex. If $\vec{a}$ = , $-\vec{a} = (-1)\vec{a}$ = , $t\vec{a}$ = ↗ if $0<t<1$,

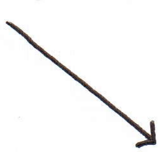
$t\vec{a}$ =  if $t>1$, and $t\vec{a}$ =  if $t<-1$

Observe that for any scalar k and vectors $\vec{a}, \vec{b}$, the vectors $k\vec{a}+k\vec{b}$ and $k(\vec{a}+\vec{b})$ are the same

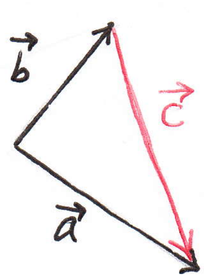


The red and black triangles are similar. Thus $k\vec{a}+k\vec{b} = k(\vec{a}+\vec{b})$ (why?)

<u>Comprehension check</u>

1) If $\vec{a}$ =  and $\vec{b}$ =  draw $\vec{a}-\vec{b}$

Solution: Draw $\vec{a}$ and $\vec{b}$ tail to tail and draw a vector with tail at the arrowhead of $\vec{b}$ and head at the arrowhead of $\vec{a}$.



Clearly $\vec{b}+\vec{c}=\vec{a}$. Thus $\vec{b}+\vec{c}-\vec{b}=\vec{a}-\vec{b}$. But by the rules of vector addition, $\vec{b}+\vec{c}-\vec{b}=\vec{b}+(\vec{c}-\vec{b})=\vec{b}+(-\vec{b}+\vec{c})=$ $=(\vec{b}-\vec{b})+\vec{c}=\vec{0}+\vec{c}=\vec{c}$. In other words, $\vec{c}$ is the desired vector.

2) Is $\|\vec{a}\|$ a vector or a number (scalar)?

Solution: $\|\vec{a}\|$ is the magnitude (length) of the vector $\vec{a}$. Thus $\|\vec{a}\|$ is a scalar.

3) Which of the following are vectors? Which are not defined?

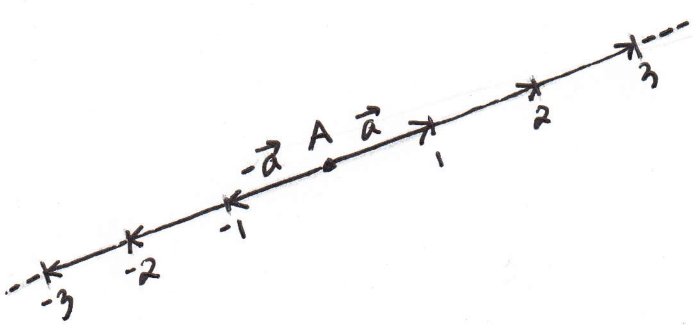
(a) $2\vec{a}+3\vec{b}$ (b) $\vec{b}-4\vec{a}$ (c) $\vec{a}\,\vec{b}$ (d) $\vec{0}\,\vec{a}$ (e) $0\,\vec{a}$

(f) $\dfrac{\vec{a}}{\|\vec{a}\|}$ (g) $\|\vec{a}\|\vec{b}-\|\vec{b}\|\vec{a}$

Solution: (a), (b), (e), and (g) are always vectors. (c) and (d) are products of vectors, which are not defined. (f) is defined with the additional assumption that $\vec{a}\neq\vec{0}$.

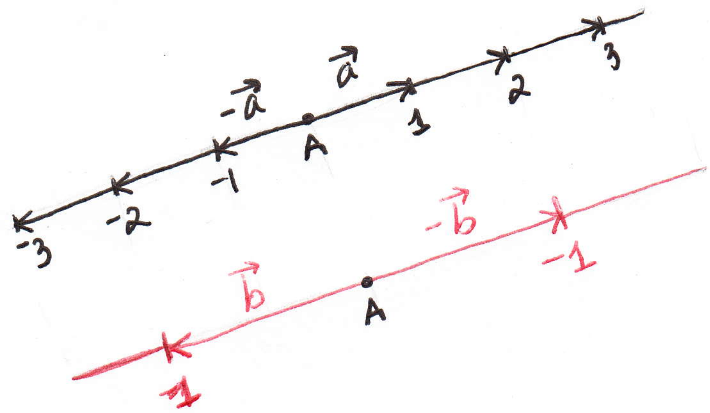
Coordinates

We have already seen that a vector $\vec{a}$, whose tail is fixed at a point A generates a number line whose "unit length" is $\|\vec{a}\|$. Each point on this number line acquires an address or coordinates in terms of $\vec{a}$



The "scalars" 1, 2, 3... represent $\vec{a}$, $2\vec{a}$, $3\vec{a}$ respectively, while "scalars" -1, -2, -3... represent $-\vec{a}$, $-2\vec{a}$, $-3\vec{a}$ respectively. The "scalar" 0 stands for the point A.

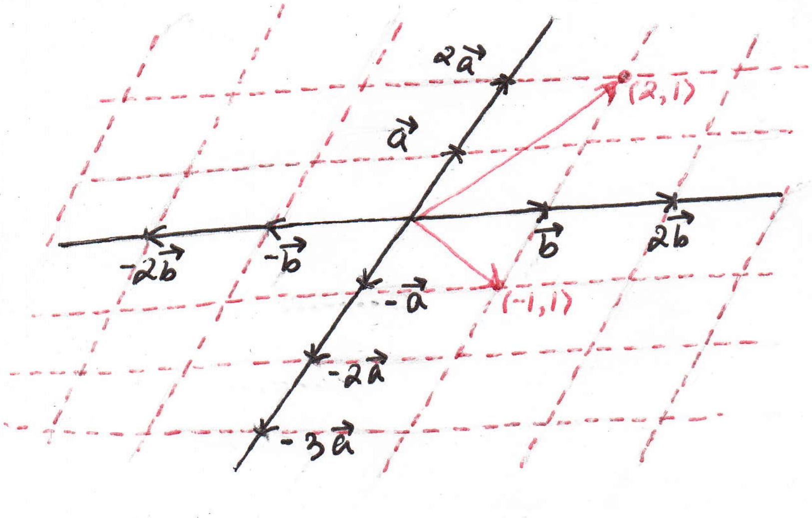
Notice that any vector $\vec{b}$ with the same or opposite direction to $\vec{a}$ generates the same line, but the coordinates $\pm 1, \pm 2, \pm 3$, etc no longer stand for the same points.



For example in the diagram above, the red coordinate 1 stands for the same point as the black coordinate -2. This suggests that $\vec{b} = -2\vec{a}$. Indeed, whenever two vectors generate the same line, one is always a scalar multiple of the other. For obvious reasons, such vectors are said to be linearly dependent.

The set of points on a line satisfies the definition of linear space. The address of each point on the line may always be given by a single real number that is uniquely paired with the point. For this reason, the line space is said to be one dimensional.

Two noncolinear vectors generate a planar grid which assigns to every point in the plane a unique coordinate pair of real numbers.

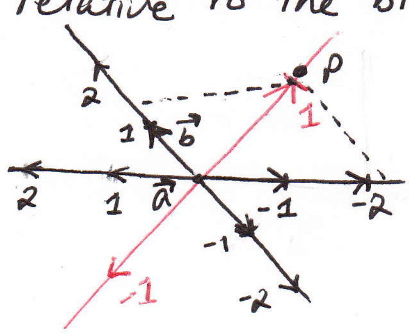


$$(s,t) \equiv s\vec{a} + t\vec{b}$$

In the planar grid above, the s-axis is generated by the vector $\vec{a}$ and the t-axis is generated by the vector $\vec{b}$. The axes are not perpendicular to each other and the "unit length" on the s-axis is different from the "unit length" on the t-axis, since $\|\vec{a}\| \neq \|\vec{b}\|$. (Think of "unit length" on the s-axis as measured in inches and the "unit length" on the t-axis as measured in centimeters).

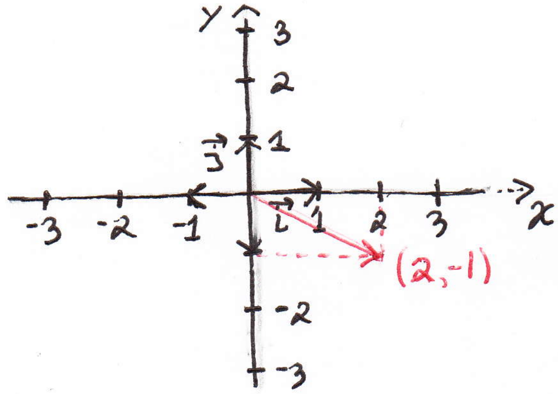
The address (s,t) corresponds to the point on the plane that we would arrive at if we displace from the center of the grid by the vector $s\vec{a}+t\vec{b}$. In other words, if you are trying to locate the point whose address is (s,t) relative to this grid system, simply move s units parallel to $\vec{a}$ and then t units parallel to $\vec{b}$. For convenience, the points whose addresses are $(-1,1)$ and $(2,1)$ relative to the coordinate grid are labled. Try to locate the points, whose coordinates are $(-2,-1)$ and $(-3,2)$.

<u>Remark</u>: It is important to distinguish points (i.e. physical locations) and coordinates. A point has a unique address relative to a fixed coordinate system. The point will have a different address relative to some other coordinate system. For example, the point P has the coordinate $(-2,1)$ relative to the black coordinate grid and 1 relative to the red number line.



The orthogonal coordinate system

It is often convenient to have a rectangular coordinate system, in which one generating vector is horizontal and the other vector is vertical and where both vectors are of the same length.
It is costumery to denote the horizontal vector by the letter $\vec{i}$ and the vertical vector by $\vec{j}$. These vectors generate the familier xy plane grid system, in which the coordinate (x,y) stands for the point $x\vec{i}+y\vec{j}$ from the origin.



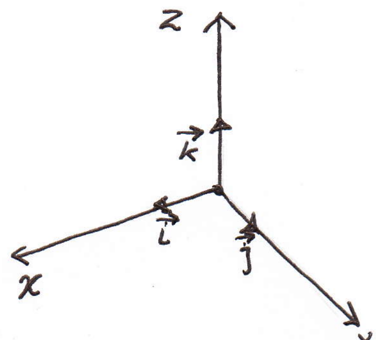
The association between, for example, the coordinate $(2,-1)$ and the point $2\vec{i}-\vec{j}$ from the origin has become so strong that we are forced to adopt the following conversion:

$\boxed{\text{(a,b) stands for the point } a\vec{i}+b\vec{j} \text{ unless another grid system is specified.}}$

Remark: The vectors $\vec{j}$ and $\vec{i}$ are sometimes denoted by the symbols $\vec{e}_2$ and $\vec{e}_1$ respectively.
With the conversion above, the set $\mathbb{R}^2 = \{(x,y) : x,y \in \mathbb{R}\}$ of coordinate pairs may be regarded as a linear space, as each coordinate is in 1-1 correspondence with some geometric point.

Coordinate systems for 3-D space

By taking any 3 vectors that do not lie in the same plane we can generate a coordinate system with 3 axes. It is particularly interesting when the generating vectors are perpendicular to each other and such that they all have the same length. In that case it is costumary to denote the axes with parameters x, y, and z and to name their generating vectors $\vec{i}, \vec{j}, \vec{k}$ (or $\vec{e_1}, \vec{e_2}, \vec{e_3}$) respectively.



The corresponding coordinate system is the set $\mathbb{R}^3 = \{(x,y,z) : x,y,z \in \mathbb{R}\}$
Thus the geometric point whose address is (x,y,z) is obtained by moving from the origin by $x\vec{i} + y\vec{j} + z\vec{k}$.
To draw a point (a,b,c) in the xyz grid

1) Lable a, b, and c on their respective axes.

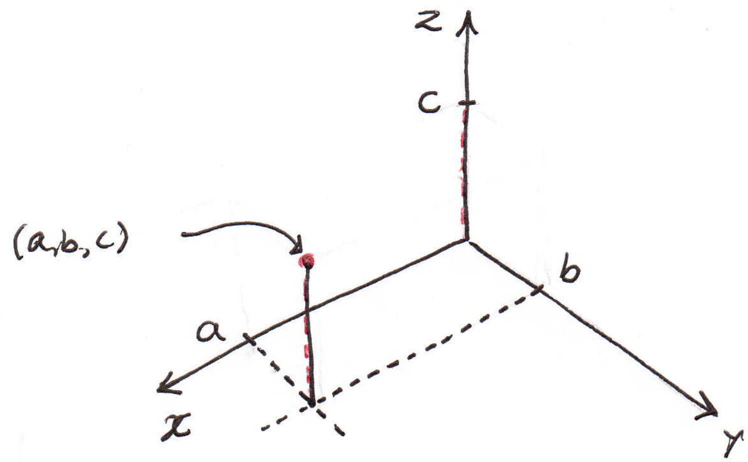
2) Draw a dotted line at $x = a$ in the xy plane parallel to the y-axis.

3) Draw a dotted line at $y = b$ in the xy plane parallel to the x-axis.

4) Mark the point of intersection of these two dotted lines.

5) Ascend by the vector $c\vec{k}$ from the point you just labled.

* Considerations of perspective are more complicated than that, but it should suffice for our purpose.



If you wish to practice, try to draw the points corresponding to $(3,0,0)$, $(0,-2,0)$, and $(2,2,4)$ in your coordinate system.

Distinguishing between Vector and point coordinates

The point whose (x,y,z) coordinate is (a,b,c) is obtained by displacing from the origin by the vector $a\vec{i}+b\vec{j}+c\vec{k}$. Thus it is natural to think of the address of a point as a vector, whose tail lies in the origin of the coordinate system.

When we wish to indicate the displacement $a\vec{i}+b\vec{j}+c\vec{k}$ as opposed to the coordinate (a,b,c), we will use the notation $\langle a,b,c\rangle$ or simply write $a\vec{i}+b\vec{j}+c\vec{k}$. For example, if $p=(P_1,P_2,P_3)$ is a point, $(P_1,P_2,P_3)+\langle a,b,c\rangle$ indicates displacement by the vector $a\vec{i}+b\vec{j}+c\vec{k}$ from the point (P_1,P_2,P_3). When the meaning is clear, we may be sloppy and write $(P_1,P_2,P_3)+(a,b,c)$ instead.

The concept of vector revisited

Earlier we defined a vector as a directed line segment. But what preceicely is direction and length? If we draw a vector $\vec{v}$ in standard position in the xyz grid (i.e with tail at the origin and arrowhead at some point (a,b,c)). Then $\vec{v}$ is described in terms of the vectors $\vec{i}, \vec{j}, \vec{k}$ as $\vec{v} = a\vec{i} + b\vec{j} + c\vec{k}$. The vector's length and direction can be preceicely described in terms of the directions and length of $\vec{i}, \vec{j}, \vec{k}$. We define the magnitude of each of the three vectors as $\|\vec{i}\| = \|\vec{j}\| = \|\vec{k}\| = 1$. We assume the directions of $\vec{i}, \vec{j}, \vec{k}$ to be understood and define the direction of $\vec{v}$ to be the vector $t\vec{v}$ such that $t > 0$ and $\|t\vec{v}\| = 1$ (The value of t will be determined later on).

Prior to this approach, we had to rely on intuition to understand the multitude of vectors in 3-D space. Now we only need to grasp three vectors. A useful consequence of this axiomatization is that we can now add and subtract vectors and multiply these vectors by scalers algebreaically rather than geometrically: Suppose $\vec{A} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ and $\vec{B} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$. Then, recalling that vector addition is commutative and associative, we get $\vec{A} + \vec{B} = (a_1\vec{i} + b_1\vec{i}) + (a_2\vec{j} + b_2\vec{j}) + (a_3\vec{k} + b_3\vec{k}) =$

$$= (a_1 + b_1)\vec{i} + (a_2 + b_2)\vec{j} + (a_3 + b_3)\vec{k}.$$

In other words, addition of vectors is done coordinatewise:

$$\langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle.$$

If $t \in \mathbb{R}$, $t\vec{A} = t(a_1\vec{i} + a_2\vec{j} + a_3\vec{k}) = (t a_1)\vec{i} + (t a_2)\vec{j} + (t a_3)\vec{k}$.

Hence $t\langle a_1, a_2, a_3 \rangle = \langle t a_1, t a_2, t a_3 \rangle$.

Ex. Let $\vec{a} = 10\vec{i} - \vec{j} + 14\vec{k}$, $\vec{b} = 6\vec{i} + 7\vec{j} - 8\vec{k}$ and $c = \frac{1}{2}$.

Calculate

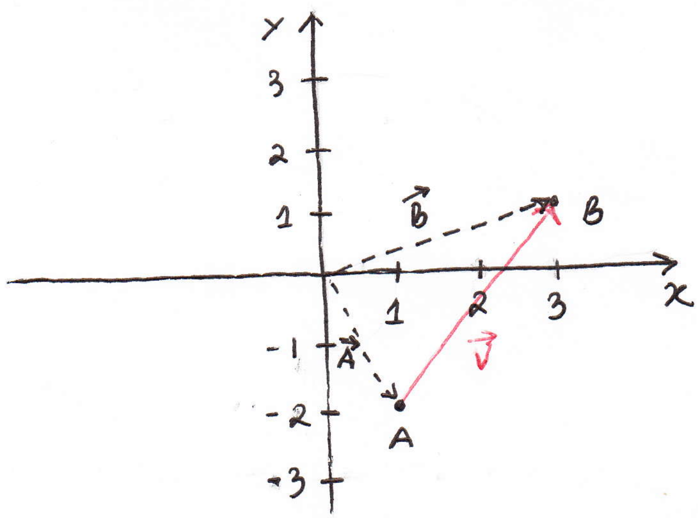
1) $\vec{a} + \vec{b}$
2) $c\vec{b}$
3) $\vec{b} - \vec{a}$

Solution:

1) $\vec{a} + \vec{b} = \langle 10, -1, 14 \rangle + \langle 6, 7, -8 \rangle = \langle 16, 6, 6 \rangle = 16\vec{i} + 6\vec{j} + 6\vec{k}$
2) $c\vec{b} = \frac{1}{2} \langle 6, 7, -8 \rangle = \langle 3, \frac{7}{2}, -4 \rangle = 3\vec{i} + \frac{7}{2}\vec{j} - 4\vec{k}$.
3) $\vec{b} - \vec{a} = \langle 6, 7, -8 \rangle - \langle 10, -1, 14 \rangle = \langle -4, 8, -22 \rangle = -4\vec{i} + 8\vec{j} - 22\vec{k}$.

Finding the algebraic expression & displacement

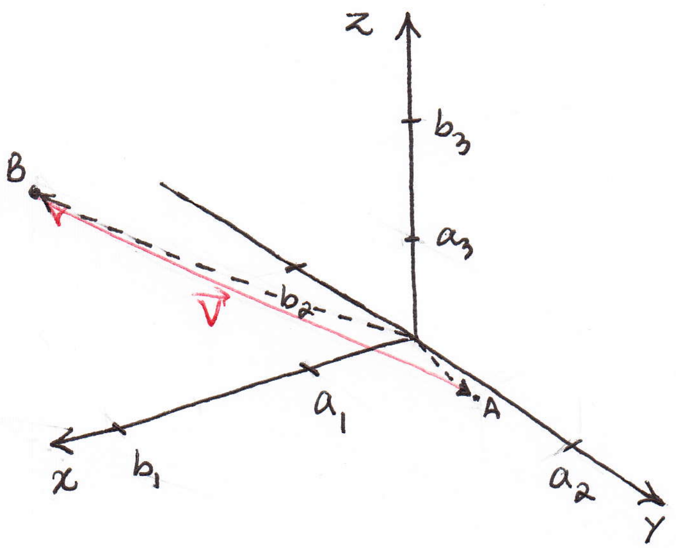
Suppose that we wish to find the displacement vector between the points $(1,-2)$ and $(3,1)$ in the xy plane



Let $\vec{v} = \langle v_1, v_2 \rangle$ be the displacement from A to B. Then $\vec{A} + \vec{v} = \vec{B}$ as seen in the diagram above. Thus

$\vec{v} = \vec{B} - \vec{A} = \langle 3,1 \rangle - \langle 1,-2 \rangle = \langle 2,3 \rangle = 2\vec{i} + 3\vec{j}$.

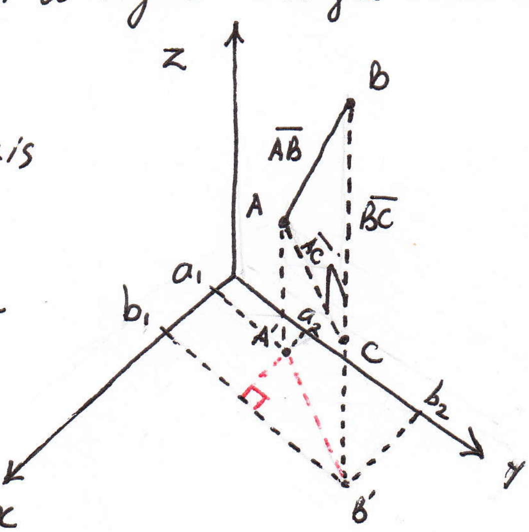
In general, if $\vec{v}$ represents the displacement from point $A = (a_1, a_2)$ to $B = (b_1, b_2)$, then $\vec{v} = \vec{B} - \vec{A} = \langle b_1 - a_1, b_2 - a_2 \rangle = (b_1 - a_1)\vec{i} + (b_2 - a_2)\vec{j}$.
Displacement in $\mathbb{R}^3$ is computed similarly. If $\vec{v}$ is a vector with tail at $A = (a_1, a_2, a_3)$ and arrowhead at $B = (b_1, b_2, b_3)$, then $\vec{v} = \langle b_1 - a_1, b_2 - a_2, b_3 - a_3 \rangle$.



Distance between points in $\mathbb{R}^3$

Let $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$ be two points in $\mathbb{R}^3$. The distance from A to B is the length of the line segment $\overline{AB}$ from A to B. The picture below showes that we may think of $\overline{AB}$ as the hypothenuse of a right triangle with vertical side $\overline{BC}$ and horizontal side $\overline{AC}$.

$\overline{BC}$ is parallel to the z-axis so its length is $|b_3 - a_3|$.
$\overline{AC}$ is parallel to $\overline{A'B'}$ in the xy plane. Its length is
$\sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2}$



By the pythagorean theorem, the length of $\overline{AB}$ is the root of the sum of squares of the triangle's sides. Thus

$$\text{Length}(\overline{AB}) = \sqrt{[\text{length}(\overline{AC})]^2 + [\text{length}(\overline{BC})]^2} = \sqrt{(b_1-a_1)^2+(b_2-a_2)^2+(b_3-a_3)^2}.$$

We also make the following definition:

Def: Let $\vec{v} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$. The norm of $\vec{v}$, $\|\vec{v}\|$, is the length of the line segment with $(0,0,0)$ and (a_1, a_2, a_3) as its endpoints. Thus, $\|\vec{v}\| = \sqrt{a_1^2+a_2^2+a_3^2}$.

If $t \in \mathbb{R}$, $t\vec{v} = (ta_1)\vec{i} + (ta_2)\vec{j} + (ta_3)\vec{k}$. Hence, $\|t\vec{v}\| =$

$$= \sqrt{(ta_1)^2+(ta_2)^2+(ta_3)^2} = \sqrt{t^2(a_1^2+a_2^2+a_3^2)} = \sqrt{t^2}\sqrt{a_1^2+a_2^2+a_3^2}$$

$$= |t|\,\|\vec{v}\|.$$

Unit vector in the direction of $\vec{v}$

Suppose we are given $\vec{v} = \langle v_1, v_2, v_3\rangle$ and wish to find a unit vector in the direction of $\vec{v}$. By definition, our vector must be of the form $t\vec{v}$, $t>0$ such that $\|t\vec{v}\| = 1$. By the work above, $1 = \|t\vec{v}\| = |t|\|\vec{v}\| = t\|\vec{v}\|$. Thus $t = \frac{1}{\|\vec{v}\|}$. Note that direction is not defined when $\vec{v} = \vec{0} = \langle 0,0,0\rangle$.

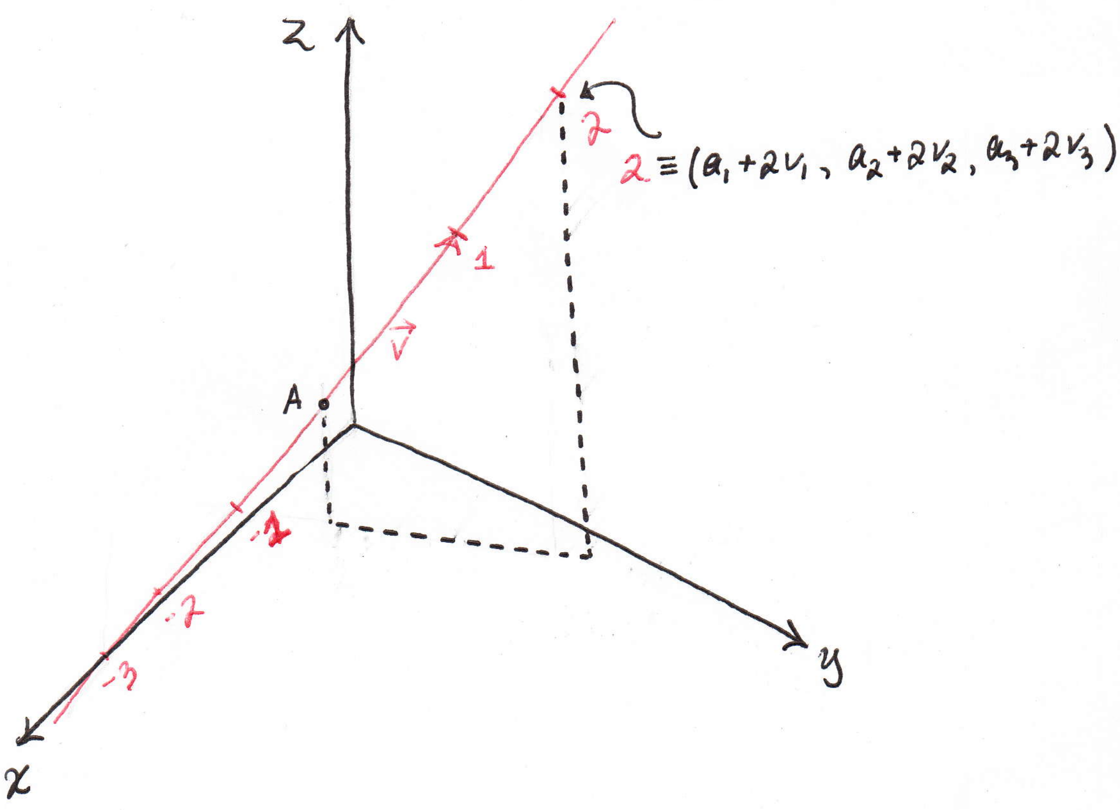
Comprehension check: Let $\vec{a} = 3\vec{i} - 4\vec{j} + \vec{k}$. Find a unit vector in the direction of $\vec{a}$. Also, find the unit vector in the opposite direction.

Solution: $\|\vec{a}\| = \sqrt{3^2+(-4)^2+1^2} = \sqrt{26}$. Hence the unit vector in the direction of $\vec{a}$ is $\frac{1}{\|\vec{a}\|}\vec{a} = \frac{1}{\sqrt{26}}(3\vec{i} - 4\vec{j} + \vec{k}) =$

$= \frac{3}{\sqrt{26}}\vec{i} - \frac{4}{\sqrt{26}}\vec{j} + \frac{1}{\sqrt{26}}\vec{k}$. The unit vector in the opposite direction is $\frac{-1}{\|\vec{a}\|}\vec{a} = \frac{-3}{\sqrt{26}}\vec{i} + \frac{4}{\sqrt{26}}\vec{j} - \frac{1}{\sqrt{26}}\vec{k}$.

Lines in $\mathbb{R}^3$

We have seen that a particle undergoing linear motion traces a number line. In particular, suppose that a particle is traveling through space in the direction of $\vec{v} = \langle v_1, v_2, v_3 \rangle$. If the particle is currently passing through the point $A = (a_1, a_2, a_3)$, each point on the number line can be assigned a one parameter address t relative to the line coordinate grid and a three parameter address relative to the xyz coordinate grid. The three parameter address is related to the one parameter address by the equation
$L(t) = A + t\vec{v} = (a_1, a_2, a_3) + t\langle v_1, v_2, v_3 \rangle = (a_1 + tv_1, a_2 + tv_2, a_3 + tv_3)$



The correspondence can also be written

$$x(t) = a_1 + tv_1$$
$$y(t) = a_2 + tv_2$$
$$z(t) = a_3 + tv_3$$

Ex. Find the equation of the line passing through $(1,0,0)$ in the direction $\vec{j}$. Is the point $(1,5,3)$ on this line?

Solution: $A=(1,0,0)$, $\vec{v}=\langle 0,1,0\rangle$. Hence $L(t)=A+t\vec{v}=(1,t,0)$

That is

$$x(t)=1$$
$$y(t)=t$$
$$z(t)=0$$

If the point $(1,5,3)$ is on the line, it satisfies the above equations.

$$1=1 \qquad (1)$$
$$5=t \qquad (2)$$
$$3=0 \qquad (3)$$

Equation (1) is satisfied for all t. Equation (2) requires $t=5$, but equation (3) is false. Therefore $(1,5,3)$ is not on the line.

Ex. 1) Find the line passing through $(3,-1,2)$ in the direction $2\vec{i}-3\vec{j}+4\vec{k}$.

2) Find the parametric equation of the line in the xy plane through $(1,-6)$ in the direction $5\vec{i}-\pi\vec{j}$.

3) In what direction does the line given by

$$x(t)=-3t+2$$
$$y(t)=-2(t-1)$$
$$z(t)=8t+2$$

point?

Solution:

1) $L(t)=(3,-1,2)+t\langle 2,-3,4\rangle=(3+2t,-1-3t,2+4t)$

2) $p(t)=(1,-6)+t\langle 5,-\pi\rangle=(1+5t,-6-\pi t)$

3) $q(t)=(-3t+2,-2[t-1],8t+2)=(2,2,2)+t\langle -3,-2,8\rangle$

Thus the line points in the direction of the vector $\langle -3,-2,8\rangle$.

Remark: Technically speaking, $L(t) = (a_1 + tv_1, a_2 + tv_2, a_3 + tv_3)$ is the description of the process by which the line gets to be traced. By speeding up or slowing down the particle that is undergoing linear motion, we still let it visit the same set of points in space. Thus, for example, if $p(t) = (a_1 + t(2v_1), a_2 + t(2v_2), a_3 + t(2v_3))$ then the particle is tracing the line with doubled speed, while $q(t) = (a_1 - tv_1, a_2 - tv_2, a_3 - tv_3)$ means that the particle traces the line in reversed direction.

Note also that the particle can trace the line with varied speed. For example, $S(t) = (a_1 + t^3 v_1, a_2 + t^3 v_2, a_3 + t^3 v_3)$ implies that the particle was slowing down in the past, stopping momentarily at the point (a_1, a_2, a_3), and speeding up as $t \to \infty$.

Def: A line parametrized by $L(t) = (a_1 + tv_1, a_2 + tv_2, a_3 + tv_3)$ is the set $L = \{L(t) : t \in \mathbb{R}\}$

Comprehension check: Describe the set of points traced by

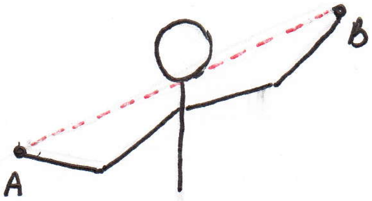
$c(t) = (1 + 3\sin t, -2 + \sin t, 5 - 2\sin t)$

Solution: $C(t) = (1, -2, 5) + \sin t \langle 3, 1, -2 \rangle$. As t varies from $-\infty$ to ∞, $\sin t$ oscilates between -1 and 1. Therefore $C(t)$ traces the line segment with endpoints $A = (1, -2, 5) - \langle 3, 1, -2 \rangle = (-2, -3, 7)$ and $B = (1, -2, 5) + \langle 3, 1, -2 \rangle = (4, -1, 3)$.

If we think of $C(t)$ as a path of a particle in time, we see that the particle goes back and forth along the line segment $\overline{AB}$, visiting each point on the line segment infinitely many times.

Finding a parametric equation of the line from two points on the line

If you hold a point in your left hand and another point in your right hand, you can readily imagine a thread stretching from the first point to the next.



This suggests that two points determine a line through space. We can parametrize this line as soon as the coordinates of the two points are available.

Let $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$. Then $\vec{v} = \vec{B} - \vec{A} =$
$= \langle b_1 - a_1, b_2 - a_2, b_3 - a_3 \rangle$ is the displacement vector from A to B.
$L(t) = A + t\vec{v}$ is a parametrization such that $L(0) = A$ and $L(1) = B$. In particular, if you think of $L(t)$ as generating a ruler, the distance between successive integers on the ruler is $\|\vec{v}\|$.
Observe that if C is another point on the line, then $C = A + t_1\vec{v}$ for some $t_1 \in \mathbb{R}$. Thus $\vec{C} - \vec{A} = t_1\vec{v}$. In other words, if $p(t) = A' + t\vec{w}$ is some other parametrization of the line going through A and B, then $A' - A$ is a scalar multiple of $\vec{v}$. Also, $\vec{w}$ is a scalar multiple of $\vec{v}$.

Ex. 1) Find a parametrization of the line that passes through the points $(1, -3, 2)$ and $(-1, 2, 0)$.

2) Do the parametrizations $L(t) = (1+t, 2+t, 3+t)$ and $p(t) = (2+t, 3+t, 4+t)$ trace the same line?

Solution:

1) Set $\vec{v} = \langle -1, 2, 0\rangle - \langle 1, -3, 2\rangle = \langle -2, 5, -2\rangle$. Then $L(t) = (1, -3, 2) + t\langle -2, 5, -2\rangle = (1-2t, -3+5t, 2-2t)$. is a valid answer.

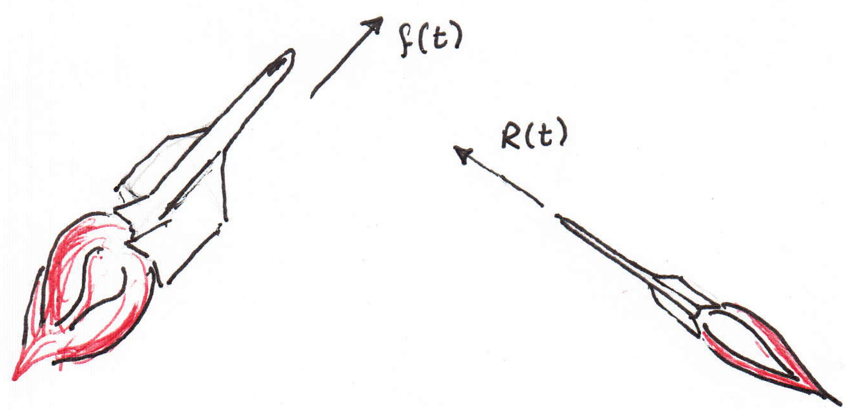
2) $L(t) = (1,2,3) + t\langle 1,1,1\rangle$, $p(t) = (2,3,4) + t\langle 1,1,1\rangle$. $(2,3,4) - (1,2,3) = \langle 1,1,1\rangle$. Therefore, $p(t)$ and $L(t)$ trace the same line.

Intersection v.s. Collision

Ex. A fighter jet is traveling in such a way that its position at time t is $f(t) = (t/2, 1, 1)$. The enemy launches a rocket to intercept the jet. The rocket is moving such that its position at time t is $R(t) = (1, t, t)$.

1) Does the rocket hit the jet?

2) Does the path of the rocket intersect the path of the jet?



Solution:

1) The rocket intercepts the jet iff the two objects pass through the same point in space at the same time. Thus, we set $R(t) = f(t)$ and check if there is a solution. The equation becomes

$$t/2 = 1 \quad (1)$$
$$1 = t \quad (2)$$
$$1 = t \quad (3)$$

These equations are inconsistent; Equation (1) is satisfied when $t=2$, while equations (2) and (3) are only valid when $t=1$. Therefore the fighter plane and the rocket do not collide.

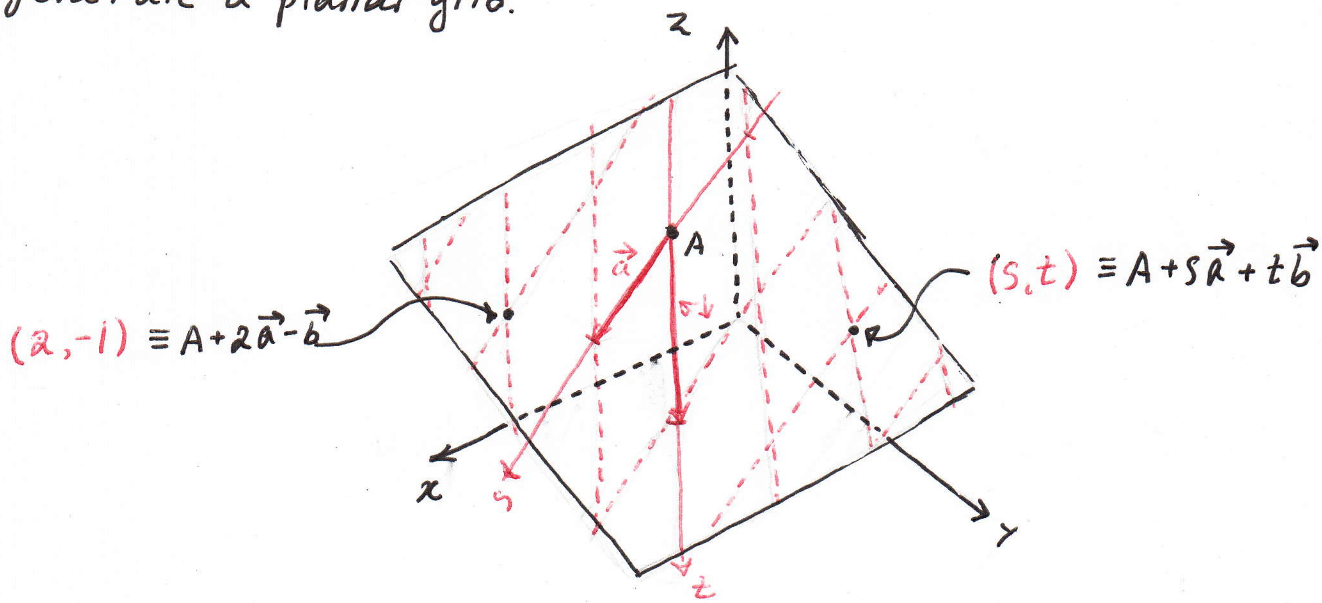
2) The paths of two objects moving through space intersect if the objects pass through the same point at (possibly) different times. Thus we must see if $f(t) = R(s)$ has a solution. Now equality holds iff

$$t/2 = 1 \qquad (1)$$
$$1 = s \qquad (2)$$
$$1 = s \qquad (3)$$

All 3 equations are satisfied when $t=2$ and $s=1$. Thus the rocket and the fighter jet pass through the point $(1,1,1)$. The rocket passes this point 1 second after it's launched. The fighter goes through the same point a second later.
Notice that $t=s$ would have implied collision.

Planes in $\mathbb{R}^3$

By fixing a point in $\mathbb{R}^3$ and two noncolinear vectors, we can generate a planar grid.



If the grid has the fixed point $A = (P_1, P_2, P_3)$ and generating vectors $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, then each point on the plane has a two parameter address (s, t) relative to the plane (indicated in red in the above diagram) and a three parameter address (x, y, z) relative to the xyz grid (indicated in black in the above diagram). The two addresses are related by the equation $P(s,t) = A + s\vec{a} + t\vec{b} =$

$$= (P_1, P_2, P_3) + s \langle a_1, a_2, a_3 \rangle + t \langle b_1, b_2, b_3 \rangle =$$

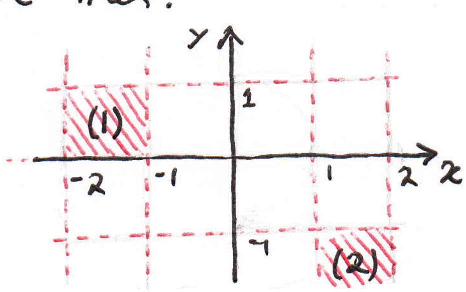
$$= (P_1 + s a_1 + t b_1,\ P_2 + s a_2 + t a_2,\ P_3 + s a_3 + t b_3).$$

Remark: If $\vec{a}$ and $\vec{b}$ are linearly dependent, that is, if $\vec{b} = r\vec{a}$ for some $r \in \mathbb{R}$, then $P(s,t)$ does not represent a parametrization of a plane. This can be seen geometrically by noting that the s and t axes would then be collapsed to the same line.

We omit the discussion on parametric equations for planes as it is similar to the one already given on lines in $\mathbb{R}^3$. The reader should consult homework set #1 for practice problems.

Parametrizations of Parallelograms

Consider the grid system generated by the vectors $\vec{i}$ and $\vec{j}$. The planar, two dimensional space that these vectors generate is neatly cut into square tiles:

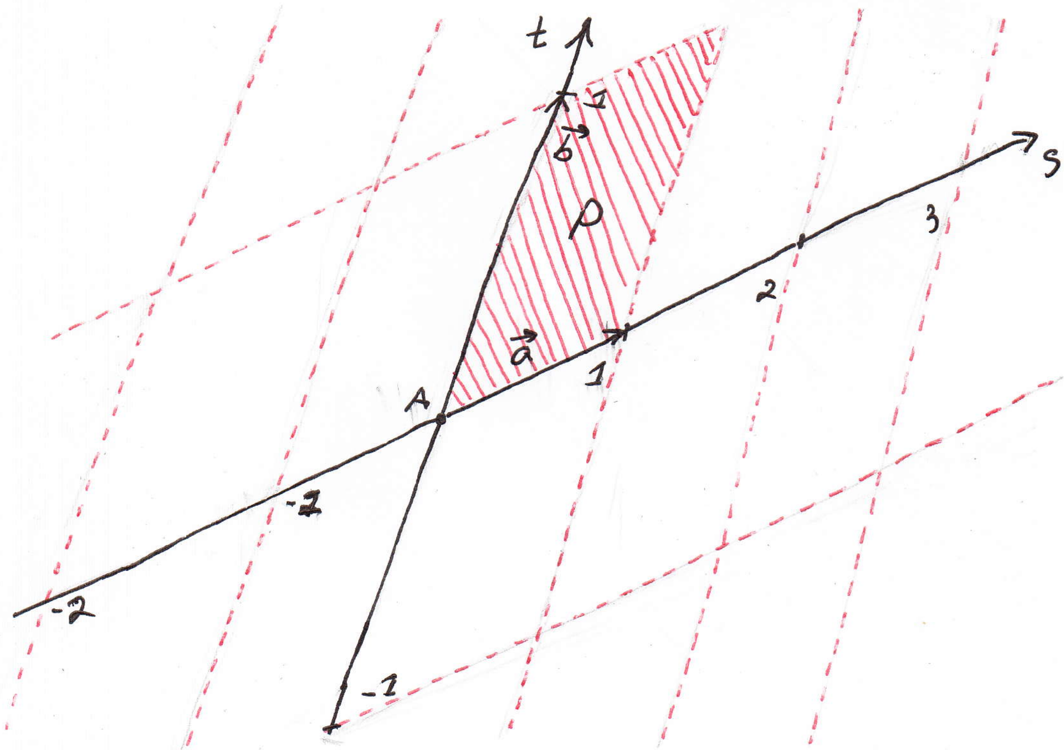


Each tile is a set of points. Tile (1), for example, is the set $\{(x,y): -2 \leq x \leq -1;\ 0 \leq y \leq 1\} = \{x\vec{i} + y\vec{j}: -2 \leq x \leq -1;\ 0 \leq y \leq 1\}$ $= \{(-2,0) + s\vec{i} + t\vec{j}: s,t \in [0,1]\}$, while tile (2) is the set $\{(x,y): 1 \leq x \leq 2;\ -2 \leq y \leq -1\} = \{(1,-2) + s\vec{i} + t\vec{j}: s,t \in [0,1]\}$

Indeed, the rectangular coordinate system is so named due to the fact that rectangular sets have the simple description

$R = \{(P_1, P_2) + s\vec{i} + t\vec{j}: s \in [a,b],\ t \in [c,d]\}$

In a slanted two dimensional coordinate grid, the sets that are most easily described are parallelograms whose sides are parallel to the axes of the grid. For example, the planar grid $P(s,t)$ on page 24 looks as follows from an appropriate angle:

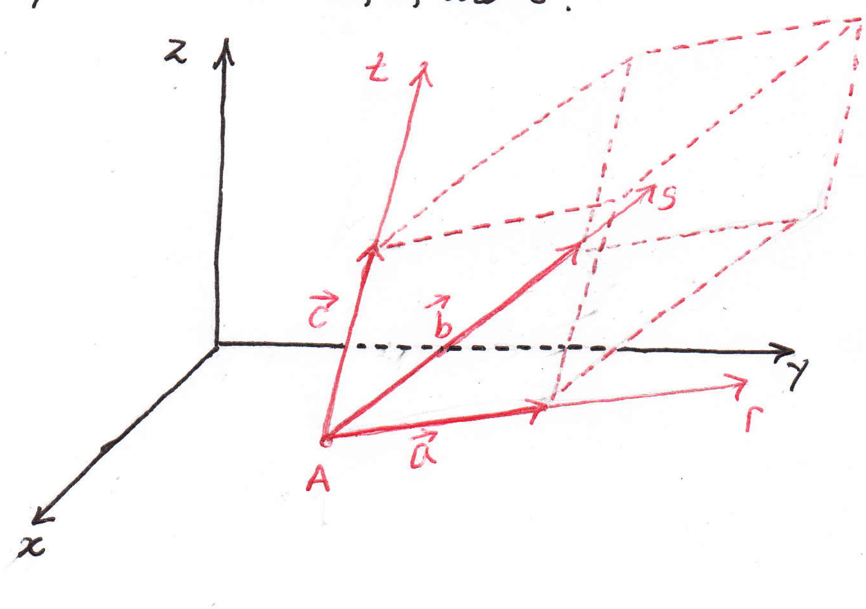


The shaded region P is given by $\{(s,t): s,t \in [0,1]\}$ in terms

of the st coordinate grid. In terms of the xyz coordinate system P is given by $\{P(s,t) : s,t \in [0,1]\} =$
$= \{A + s\vec{a} + t\vec{b} : s,t \in [0,1]\}$

Def: If $A = (P_1, P_2, P_3)$ is a point in $\mathbb{R}^3$, $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ and $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$, then $P = \{A + s\vec{a} + t\vec{b} ; s,t \in [0,1]\}$ represents the parallelogram with corner at A and edges of size $\|\vec{a}\|$ and $\|\vec{b}\|$ along $\vec{a}$ and $\vec{b}$ respectively.
The vectors $\vec{a}$ and $\vec{b}$ are said to span the parallelogram.

In an analogous manner, an orthogonal 3-D coordinate system tesselates $\mathbb{R}^3$ into unit cubes, while a slanted 3-D coordinate system cuts the space into three dimensional analogues of parallelograms called parallelepipeds.
More preceicely, if A is a fixed point (P_1, P_2, P_3) relative to the xyz grid, $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$, $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$, and $\vec{c} = c_1\vec{i} + c_2\vec{j} + c_3\vec{k}$ are vectors that do not lie on the same plane, then the set $\{A + r\vec{a} + s\vec{b} + t\vec{c} : r,s,t \in [0,1]\}$ is a parallelepiped with corner at A that is spanned by the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$.



Purpose

Many objects of practical interest are very complicated. It is difficult enough to master the mathematical properties of simple shapes like lines, planes, and parallelepipeds. How hopelessly dense must real world mathematics be? The genious of calculus is in my opinion summed up in the simple idea that the complicated can be built block by block from the simple in the manner a castle can be built from lego blocks or an image is constructed from pixels. The planes, lines, parallelograms, and parallelepipeds will be our pixels, each in its appropriate context. We illustrate this idea with the following, somewhat whimsical example.

Ex. (why King-kong cannot be big enough for Hollywood?)

King-Kong is a giant ape, apparently a scaled up version of a gorilla. Now the gorilla is made up of cells. Since each cell is tiny, we can approximate it with a tiny cube of side length r. The cells are attached to one another with the help of collagen, which acts like glue; the adhesive strength of a cell to its neighbours is proportional to the surface area of the walls of the cell in contact with the walls of other cells. Thus, if we think of the cell as a cube, the force required to detach the cell from its environment should not exceed $6kr^2$, where 6 is the number of faces of the cube, r^2 is the surface area of the cube, and k is the constant of proportionality.

The adhesive force must be stronger than the force of gravity on the cell to prevent it from falling off due to its weight (and the weight of other cells that it supports). If ρ is the density of the cell, the force on the cell due to gravity is at least $g\rho r^3$, where g is the gravitational acceleration.

As a cell grows larger in size so do its adhesive strength and weight. However,

$$\lim_{r \to \infty} \frac{6 k r^2}{g \rho r^3} = \lim_{r \to \infty} \frac{6 k}{g \rho r} = 0$$

Thus, the weight of the cell would eventually be too great for the adhesive force to support. This means that if we make a scaled-up version of a gorilla, the animal will simply disintegrate.